

Phantom Wormhole Solutions in a Generic Cosmological Constant Background

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There are a number of reasons to study wormholes with generic cosmological constant Λ . Recent observations indicate that present accelerating expansion of the universe demands $\Lambda > 0$. On the other hand, some extended theories of gravitation such as supergravity and superstring theories posses vacuum states with $\Lambda < 0$. Even within the framework of general relativity, a negative cosmological constant permits black holes with horizons topologically different from the usual spherical ones. These solutions are convertible to wormhole solutions by adding some exotic matter. In this paper, the asymptotically flat wormhole solutions in a generic cosmological constant background are studied. By constructing a specific class of shape functions, mass function, energy density and pressure profiles which support such a geometry are obtained. It is shown that for having such a geometry, the wormhole throat r_0 , the cosmological constant Λ and the equation of state parameter ω should satisfy two specific conditions. The possibility of setting different values for the parameters of the model helps us to find exact solutions for the metric functions, mass functions and energy-momentum profiles. At last, the volume integral quantifier, which provides useful information about the total amount of energy condition violating matter is discussed briefly.

I. INTRODUCTION

In general relativity, geometrical bridges connecting two distant regions of a universe or even two different universes are in principle possible. Spacetimes containing such bridges appear as solutions of the Einstein field equations. The term "wormhole" for these bridges was used for the first time in 1957 by J. A. Wheeler [1, 2]. Many years later in 1988, the notion of traversable Lorentzian wormholes attracted the attention of physicists by the fundamental papers of Morris, Thorne and Yurtsewer [3, 4]. In these papers it was shown that such wormholes could allow humans not only to travel between universes, or distant parts of the same universe, but also to construct time machines. Also, It has been suggested that black holes and wormholes are interconvertible structures and stationary wormholes could be possible as final states of black-hole evaporation [5]. Moreover, it is shown that astrophysical accretion of ordinary matter could convert wormholes into black holes [6–8]. In the wormhole physics, it is known that these structures do not satisfy common energy conditions. The energy-momentum tensor of the matter supporting such geometries violates the null energy condition at least in the vicinity of the wormhole throat [9–11]. The matter that violates the null energy condition is usually called as exotic matter. Since the violation of the energy conditions is conventionally considered as a problematic issue, minimizing its usage seems to be useful. One may obtain that in the context of thin-shell wormholes using the cut-and-paste procedure [12, 13]. In this context, the exotic matter is concentrated at the throat of the wormhole, which is localized on the thin shell.

Another approach lies within modified theories of gravity, where normal matter threading the wormhole satisfies the energy conditions, and they are the higher order curvature terms that support these exotic geometries. In the context of modified gravity, the gravitational field equation may be written as $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu}^{eff}$, where $T_{\mu\nu}^{eff}$ is the effective energy-momentum tensor. Then, in modified theories of gravity, the effective energy-momentum tensor involving higher order derivatives violates the null energy condition, i.e., $T_{\mu\nu}^{eff}k^\mu k^\nu < 0$ where k^μ is a null vector. This approach is widely analysed in the literature as in the fame work of $f(R)$ gravity [14, 15], curvature matter couplings [16, 17], conformal Weyl gravity [18] and braneworlds [19]. On the other hand, according to the recent discoveries in cosmology, our universe is in accelerated expansion [20–23]. A dominating dark energy component with an equation of state $p = \omega\rho$ with $\omega < -\frac{1}{3}$, is thought to be responsible for this accelerated expansion phase of universe. The specific ranges of $\omega < -1$, $\omega = -1$ and $-1 < \omega < -1/3$ correspond to the phantom energy, cosmological constant and quintessence matter, respectively. Then, one of the reasons to study wormholes with generic cosmological constant Λ and specially $\Lambda > 0$ turns to the accelerated expansion of the universe. Another reason to investigate wormhole solutions with generic cosmological constant Λ turns to supergravity and superstring theories which have vacuum

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states with $\Lambda < 0$. Also, in the framework of general relativity, a negative cosmological constant allows black hole solutions with horizons that are topologically different from the usual spherical ones. Adding some exotic matter can convert these black hole solutions to wormhole solutions [24, 25].

In addition, the phantom energy possess some special features such as a divergent cosmic scale factor in a finite time [26, 27], leading to appearance of negative entropy and temperature [28, 29] and predicting a new long range force [30]. Since the fundamental ingredient of wormhole geometries is the null energy condition violation, phantom energy can provide a means to support traversable wormhole geometries [31–33]. Indeed, due to the acceleration of the universe, it seems possible that macroscopic wormholes naturally grow from submicroscopic states that originally pervaded the quantum foam. Moreover, it could be imagined an absurdly advanced civilization mining the cosmic fluid for phantom energy necessary to construct and sustain a traversable wormhole [32, 33]. Another point is that as the phantom energy equation of state represents a spatially homogeneous cosmic fluid and is assumed not to cluster, it is also possible that inhomogeneities may arise due to gravitational instabilities. Thus, density fluctuations in the cosmological background may be the origin of phantom wormholes. It can also be considered that these structures are sustained by their own quantum fluctuations [34–36].

Many of the papers published on the phantom energy wormholes are not asymptotically flat [31–33]. The approach of these papers is to glue the interior wormhole metric to a vacuum exterior spacetime at a junction interface [37–41]. Recently, new asymptotically flat phantom wormhole solutions with no need to surgically pasting the interior wormhole geometry to exterior vacuum spacetime have been found in [42]. On the other hand, spherically symmetric and static traversable Morris-Thorne wormholes in the presence of a generic cosmological constant Λ are analyzed in [43]. In that paper, two spacetimes are glued into each other and explored under matching conditions for the interior and exterior spacetimes. Another paper in this direction is [44] in which the cosmological constant is considered as a space variable scalar ($\Lambda = \Lambda(r)$).

In order to study the effects of the cosmological constant background on the asymptotically flat wormholes, we use a special class of wormholes solutions introduced by Lobo et al. [42] which does not need the cut and paste procedure. We embed these asymptotically flat wormholes into a generic cosmological constant background rather than a vacuum spacetime. Indeed, there are two inner and outer spacetimes like as the ones in the cut and paste procedure. But in this work and in contrast to the cut and paste procedure, an inner asymptotically vanishing geometry (asymptotically flat wormhole) is smoothly switching to the outer de Sitter or anti-de Sitter spacetime without need to a surgically pasting. Near the throat, the wormhole geometry is dominant while as the radii increases the wormhole features disappear and the characteristic of the cosmological constant will be more clarified, signalling us that our solutions include an asymptotic spacetime, the background, which should normally be de Sitter or anti-de Sitter, depending on the positive or negative values of the cosmological constant. It is shown that for having such a geometry, the wormhole throat r_0 , cosmological constant Λ and the equation of state parameter ω must satisfy certain conditions. The organization of this paper is as follows: In section II, general geometries and constraints of Lorentzian wormholes are outlined. In section III, Einstein field equations and the metric functions are studied and the above mentioned conditions on r_0 , Λ and ω are obtained. In sections IV and V, some specific solutions with their mass function and energy-momentum tensor profiles are presented. At the end of section V, the volume integral quantifier, for general solutions obtained in section III, is briefly mentioned. Finally, in section VI, we present our concluding remarks. Throughout this work, units of $G = c = 1$ are used.

II. GENERAL GEOMETRY AND CONSTRAINTS OF LORENTZIAN WORMHOLES

The general static and spherically symmetric Lorentzian wormhole metric is given by

$$ds^2 = -U(r)dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\Omega^2, \quad (1)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. The metric functions $U(r)$ and $b(r)$ are referred to as the redshift function and shape function, respectively. The general constraints on the redshift and shape functions which build up a wormhole are as follow:

1. The wormhole throat, which connects two asymptotic regions, is located at the minimum radial coordinate r_0 at which $b(r_0) = r_0$.
2. The shape function $b(r)$ must satisfy the so-called *flaring-out condition*, which is valid at or near the throat vicinity, given by

$$\frac{b(r) - rb'(r)}{2b^2(r)} > 0, \quad (2)$$

which at the throat of the wormhole reduces to $b'(r_0) < 1$.

3. In order to keep the proper signature of the metric, for the radial coordinates $r > r_0$, the shape function should satisfy the condition

$$1 - \frac{b(r)}{r} > 0. \quad (3)$$

4. In order to have asymptotically flat geometries, the metric functions need to obey the following conditions at $r \rightarrow \infty$:

$$\begin{aligned} U(r) &\rightarrow 1, \\ \frac{b(r)}{r} &\rightarrow 0. \end{aligned} \quad (4)$$

Obviously, these conditions may be relaxed for no-asymptotically flat wormholes.

5. To ensure the absence of horizons and singularities, it is also required that $U(r)$ be finite and nonzero throughout the spacetime.

Notice that the above constraints provide a minimum set of conditions which is mandatory for characterizing the geometry of two asymptotically flat regions connected by a bridge [45].

III. EINSTEIN FIELD EQUATIONS AND THE METRIC FUNCTIONS

We consider an anisotropic fluid for the matter content of the spacetime in the form of $T^\mu_\nu = \text{diag}(-\rho, p_r, p_l, p_l)$ where $\rho(r)$ represents the energy density, $p_r(r)$ is the radial pressure and $p_l(r)$ stands for the lateral pressure measured in the orthogonal direction to the radial direction. The Einstein equation with the cosmological constant Λ

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \quad (5)$$

leads to the following equations

$$b'(r) = (8\pi\rho(r) - \Lambda)r^2, \quad (6)$$

$$\frac{U'(r)}{U(r)} = \frac{8\pi p_r(r)r^3 + \Lambda r^3 + b(r)}{r(r - b(r))}, \quad (7)$$

$$p_l(r) = p_r(r) + \frac{r}{2} \left[p'_r(r) + (\rho(r) + p_r(r)) \frac{U'(r)}{2U(r)} \right] \quad (8)$$

where the prime sign denotes the derivative with respect to the radial coordinate r . From equation (6), it is apparent that the total density $\rho(r)$ is as $\rho(r) = \rho_w(r) + \rho_\Lambda$ where $\rho_w(r) = \frac{b'(r)}{8\pi r^2}$ is the density profile induced by the wormhole structure and $\rho_\Lambda \equiv \frac{\Lambda}{8\pi}$ is the density of the cosmological constant Λ which can be either positive or negative representing the de Sitter and anti-de Sitter regimes, respectively.

One can define a mass function $m(r)$ according to

$$m(r) \equiv \int_{r_0}^r 4\pi r^2 \rho(r) dr, \quad (9)$$

This equation together with the equation (6) leads to

$$m(r) = \frac{1}{2} \left[b(r) - r_0 + \frac{\Lambda}{3}(r^3 - r_0^3) \right], \quad (10)$$

which clearly vanishes at the wormhole throat $r = r_0$. Indeed, the inclusion of the cosmological constant will shift the respective values of $\rho(r)$, $p_r(r)$ and $p_l(r)$ and might help in minimizing the amount of energy condition violating matter which can be seen directly from equation (10). Although the corresponding mass of the cosmological constant

is unbounded, the wormhole part may or may not be bounded. We will consider these two possibilities in coming next sections.

In this paper, we are interested in the wormhole solutions using the barotropic equation of state $p_r(r) = \omega\rho(r)$. Thus, using equations (6) and (7) we obtain

$$\frac{U'(r)}{U(r)} = \frac{r\omega b'(r) + b(r) + \Lambda(1 + \omega)r^3}{r(r - b(r))}. \quad (11)$$

Before going further, it is important to point out a subtlety in considering the phantom energy equation of state in inhomogeneous spherically symmetric wormhole spacetimes. As emphasized in [31] and [32, 33], the phantom dark energy is a homogeneously distributed fluid with an isotropic pressure. However, it can be extended to the context of inhomogeneous spacetimes by considering a negative radial pressure in the equation of state. Then, the lateral pressure can be obtained using equation (8) which is coming from the Einstein field equations. This approach is motivated by the discussion of the inhomogeneities that may appear because of gravitational instabilities and through the analysis carried out in [46]. The authors of [46] investigated a spherically symmetric time dependent wormhole solution in a cosmological context with a ghost scalar field. As a result, the radial pressure is negative through the spacetime and for large values of the radial coordinate equals to the lateral pressure, which shows the behavior of ghost scalar field as dark energy.

We have now four equations, equations (6)-(8) and (11), with five unknown quantities $U(r)$, $b(r)$, $\rho(r)$, $p_r(r)$ and $p_l(r)$. There are two different approaches for solving the field equations. One approach is to consider a specific distribution of the energy density threading the wormhole, like the approach of [31] and consequently finding the metric functions $U(r)$ and $b(r)$. The second approach involves proposing a model wormhole geometry by imposing specific choices for the shape and redshift functions and obtaining the supporting energy-momentum tensor profile [32, 33]. In this paper, since we are preliminary interested in finding asymptotically de Sitter and anti- de Sitter wormhole solutions, which are supported by phantom or non-phantom matter contents, the second approach is followed.

We consider wormholes with the shape function

$$\frac{b(r)}{r_0} = a \left(\frac{r}{r_0} \right)^\alpha + C, \quad (12)$$

where a , α and C are dimensionless constants. The first constraint imposes that $C = 1 - a$. In order to satisfy the fourth constraint, we get $\alpha < 1$. Thus, the shape function takes the form

$$b(r) = r_0 + ar_0 \left[\left(\frac{r}{r_0} \right)^\alpha - 1 \right]. \quad (13)$$

In order to satisfy the flaring out condition we obtain

$$1 - a + a \left(\frac{r}{r_0} \right)^\alpha (1 - \alpha) > 0. \quad (14)$$

Form the previously obtained result, $\alpha < 1$, we may divide the solutions of this condition into the following cases: i) $0 \leq a \leq 1$ which clearly satisfies the above condition, ii) $a > 1$ whose exact value depends on α and r and iii) $a < 0$ which similar to the previous case, its exact value depending on α and r . In addition, these three classes should also satisfy the condition $a\alpha < 1$ coming from the flaring out condition at the throat.

Using equations (6) and (13), we can obtain the total energy density $\rho(r)$ as

$$\rho(r) = \frac{1}{8\pi} \left(\frac{\alpha a}{r_0^2} \left(\frac{r}{r_0} \right)^{\alpha-3} + \Lambda \right), \quad (15)$$

where for the case of $\Lambda = 0$, the profile density of [42] are recovered. Also, from equation (15), it is seen that in order to obtain de Sitter or anti-de Sitter solutions when $r \rightarrow \infty$ we should have $\alpha < 3$. Then, our obtained restricted regime $\alpha < 1$ includes these asymptotic behaviors.

Also, the total energy density $\rho(r)$, equation (15), should satisfy the positive energy condition

$$\frac{\alpha a}{r_0^2} \left(\frac{r}{r_0} \right)^{\alpha-3} + \Lambda \geq 0, \quad (16)$$

which is valid for $\Lambda > 0$ with $\alpha a \geq 0$. Then, with respect to the above three classes of a values and the condition $a\alpha < 1$ coming from the flaring out condition at the throat, we will have $0 \leq \alpha a < 1$ and the following classes are

distinguished: i) $\Lambda > 0$ with ranges of $0 \leq a \leq 1$ and $0 \leq \alpha < 1$, ii) $\Lambda > 0$ with ranges of $1 < a$ and $0 \leq \alpha < 1$ and iii) $\Lambda > 0$ with ranges of $a < 0$ and $\alpha \leq 0$.

The condition (16) can be satisfied for $\Lambda < 0$ with $0 < \alpha a < 1$ and $\Lambda > 0$ with $\alpha a < 0$, but these cases are completely dependent on exact numerical value of Λ . The case of $\alpha a < 0$ is arising from the presence of a positive cosmological constant Λ which is not allowed in the absence of Λ as in [42]. Then, in the presence of the cosmological constant, we have an extended class of the asymptotically flat wormhole solutions than the ones in the absence of it. In addition, when $\Lambda < 0$, the positive energy condition for the energy density $\rho(r) = \rho_w(r) + \rho_\Lambda \geq 0$ can be violated. The same situation occurs in the anti de-Sitter spacetime [47].

Returning to the equation (16), we can obtain a condition for the throat of the wormhole, $r = r_0$, when $\Lambda > 0$ with the range of $0 \leq \alpha a < 1$ as

$$r_0^2 \geq -\frac{\alpha a}{\Lambda}, \quad (17)$$

where it is trivial and does not put any restriction on the size of the throat r_0 . For the case of $\Lambda > 0$ with $\alpha a < 0$, we also recover this condition but since αa has negative values, it will be a nontrivial restriction on the size of wormhole throat. If we consider $\Lambda < 0$, we obtain

$$r_0^2 \leq -\frac{\alpha a}{\Lambda}, \quad (18)$$

where is a nontrivial restriction on throat size, since for this case we should have just $0 < \alpha a < 1$. Equations (17) and (18) reveal the dependence of the size of the wormhole throat on the cosmological constant Λ and shape function parameters a and α . In the absence of the cosmological constant, one may not deduce such a direct result about the wormholes throat size in terms of its characterizing parameters a and α in equation (13).

The condition of an event-horizon-free spacetime requires that $U(r)$ to be finite and nonzero. Thus, due to the finiteness of $U(r)$, as seen by equation (7), the radial pressure evaluated at throat should be

$$p_r(r_0) = -\frac{1}{8\pi} \left(\Lambda + \frac{1}{r_0^2} \right), \quad (19)$$

Similar situation arises also in the absence of cosmological constant, as is shown below equation (15) in reference [42]. Then, we have a negative pressure at the throat which provides the geometry with a repulsive character, preventing the wormhole throat from collapsing. This characteristic is also exist for a positive cosmological constant. For a negative cosmological constant, in order to have negative pressure at the throat, we need $r_0^2 < -\frac{1}{\Lambda}$ which can be consistent with the obtained tighter constraint (18) coming from the positive energy condition. In fact, the acceptable common range which satisfies both of these conditions, is as equation (18). Equation (19) together with the equation of state $p_r(r) = \omega \rho(r)$ leads to

$$\rho(r_0) = -\frac{1}{8\pi\omega} \left(\Lambda + \frac{1}{r_0^2} \right). \quad (20)$$

On the other hand, by substituting $r = r_0$ in equation (12) we obtain

$$\rho(r_0) = \frac{1}{8\pi} \left(\frac{\alpha a}{r_0^2} + \Lambda \right), \quad (21)$$

where the consistency between the two above equations gives the cosmological constant as

$$\Lambda = -\frac{1 + \alpha a \omega}{r_0^2 (1 + \omega)}. \quad (22)$$

In the absence of Λ as in [42], for the considered shape function (13), the supporting matter with $\omega = -\frac{1}{\alpha a}$ lies just in the phantom area with no lower bound. Thus, in order to obtain solutions for $\Lambda > 0$ with $0 \leq \alpha a < 1$, we should have $-\frac{1}{\alpha a} < \omega < -1$ which points to a restricted phantom era with a lower bound specified by given α and a values. Also, when we have $\Lambda < 0$ and $0 < \alpha a < 1$, there are accessible solutions for both ranges $\omega < -\frac{1}{\alpha a} < -1$ and $\omega > -1$ which correspond the phantom and non-phantom regimes, respectively. The solutions with non-phantom regimes will be naturally ruled out by the flaring out condition. It is well known that the flaring out condition, equation (2), leads to $\rho + p_r < 0$ for the fluid near a wormhole throat which demands the matter fields with $\omega < -1$, the phantom matters. This is the well-known violation of the null and weak energy conditions. Equation (22) gives a direct constraint for

the throat size of the wormholes living in a generic cosmological background. For given values of the cosmological constant, the shape function characterizing parameters a and α with equation of state parameter of the wormhole supporting matters ω , the throat size will be fixed. For a wormhole embedded in a vacuum spacetime, one can not deduce such a direct constraint on its throat size.

Considering the shape function given by equation (13), the ordinary differential equation for the redshift function (11) takes the following form

$$\frac{U'(r)}{U(r)} = \frac{\Lambda(1+\omega)r}{1 - \left(\frac{r_0}{r}\right) \left(1 + a \left[\left(\frac{r}{r_0}\right)^\alpha - 1\right]\right)} + \left(\frac{r_0}{r^2}\right) \frac{1 + a \left[\left(\frac{r}{r_0}\right)^\alpha (1 + \alpha\omega) - 1\right]}{1 - \left(\frac{r_0}{r}\right) \left(1 + a \left[\left(\frac{r}{r_0}\right)^\alpha - 1\right]\right)}, \quad (23)$$

where the results of [42] are simply recovered by substituting $\Lambda = 0$. On the other hand, equations (13) and (15) can be used to write the shape function as

$$\begin{aligned} b(r) &= \frac{8\pi}{\alpha} \rho(r) r^3 - \frac{\Lambda}{\alpha} r^3 + r_0(1-a) \\ &= \frac{8\pi}{\alpha\omega} p_r(r) r^3 - \frac{\Lambda}{\alpha} r^3 + r_0(1-a). \end{aligned} \quad (24)$$

It is seen that this equation reveals the de Sitter or anti-de Sitter background of the whole spacetime. Note that equation (24) does not contradict with the asymptotically flatness of the wormhole geometry, because of the matter density obtained in equation (15). In fact, the considered asymptotic flat form for the shape function, equation (13), affects the amount of energy condition violating matter and couples the wormhole throat size r_0 , cosmological constant Λ and equation of state parameter ω to each other as obtained in equations (17), (18) and (22).

Consequently, one is able to substitute equation (22) into equation (11) and after using equation (6), obtain the following

$$\frac{U'(r)}{U(r)} = \frac{8\pi p_r \left(\frac{1+\alpha\omega}{\alpha\omega}\right) r^3 + \Lambda \left(\frac{\alpha-1}{\alpha}\right) r^3 + r_0(1-a)}{r^2 \left(1 - \frac{8\pi}{\alpha\omega} p_r r^2 + \frac{\Lambda}{\alpha} r^2 - \frac{r_0}{r} (1-a)\right)}. \quad (25)$$

Unfortunately, equations (23) and (25) in general have not an exact solution. Thus, in order to deduce exact wormhole solutions for this equations, we will consider some specific choices for the parameters a and α in the next sections.

Meanwhile, using equations (24) and (25) we can rewrite the lateral pressure (8) in a general form as

$$\begin{aligned} p_l(r) &= p_r(r) \left(\frac{\alpha-1}{2}\right) + \frac{\Lambda\omega(3-\alpha)}{16\pi} \\ &+ p_r(r) \left(\frac{1-\alpha a}{4(\Lambda r_0^2 + 1)}\right) \frac{8\pi p_r(r) \left(\frac{\alpha\omega+1}{\alpha\omega}\right) r^2 + \Lambda \left(\frac{\alpha-1}{\alpha}\right) r^2 + \frac{r_0}{r} (1-a)}{1 - \frac{8\pi}{\alpha\omega} p_r(r) r^2 + \frac{\Lambda}{\alpha} r^2 - \frac{r_0}{r} (1-a)}, \end{aligned} \quad (26)$$

where the first and second terms are due to the pure mass and pure cosmological constant effects, respectively, while the third term is a mixed term.

Obviously, in the presence of cosmological constant, the energy density profile $\rho(r)$, equation (15), have an asymptotically de Sitter or anti de Sitter behavior as $r \rightarrow \infty$. Thus, the radial pressure will take an asymptotically de Sitter or anti de Sitter behavior $p_r \rightarrow \frac{\omega\Lambda}{8\pi}$ as $r \rightarrow \infty$ where ω should be -1 at spatial infinity. For the lateral pressure, the second term in brackets in equation (8) with respect to equations (15) and (25) will vanish at $r \rightarrow \infty$. Then, the lateral pressure also will take an asymptotically de Sitter or anti de Sitter behavior $p_l \rightarrow \frac{\omega\Lambda}{8\pi}$ as $r \rightarrow \infty$ where ω should be -1 at spatial infinity.

Also, one may consider a constant redshift function for simplicity in which case the de Sitter or anti-de Sitter asymptotics are simply achieved [48]. Since any constant redshift function can be absorbed in the re-scaled time coordinate, we will consider $U(r) = 1$. This consideration along with Eq. (13), guarantees the asymptotically flatness condition for the inner spacetime (the wormhole spacetime). Finally, we recover the field equations as follows

$$\frac{b'(r)}{r^2} = 8\pi\rho(r) - \Lambda, \quad (27)$$

$$-\frac{b}{r^3} = 8\pi p_r(r) + \Lambda, \quad (28)$$

$$\frac{b - b'r}{2r^3} = 8\pi p_t(r) + \Lambda. \quad (29)$$

Clearly, using the shape function $b(r)$, equation (13), we have de Sitter or anti-de Sitter asymptotics as $\rho \rightarrow \frac{\Lambda}{8\pi}$, $p_r \rightarrow -\frac{\Lambda}{8\pi}$ and $p_t \rightarrow -\frac{\Lambda}{8\pi}$ as $r \rightarrow \infty$. Also, for the case of constant redshift function, one may go further and consider equation of state $p_r = \omega\rho$ and using equations (27) and (28) obtains the space varying equation of state parameter ω as

$$\omega(r) = -\frac{\Lambda r^3 + b}{\Lambda r^3 + r b'}, \quad (30)$$

which help us to achieve a better understanding of the behavior of the dominant fluid and reveals de Sitter or anti-de Sitter nature of spacetime as $\omega \rightarrow -1$ as $r \rightarrow \infty$. This equation leads to equation (22) at the throat of the wormhole and shows that ω as a space varying parameter is not allowed to be $\omega = -1$ at the throat of the wormhole. The generalization of equation (30) to the case of $U(r) \neq \text{constant}$ is also applicable by using equations (6) and (7).

As we saw, while cosmological constant affects the wormhole and its properties, the wormhole geometry will disappear in the $r \rightarrow \infty$ limit. We should also note that the asymptotically flatness condition for the inner spacetime leads to $g_{\alpha\beta} \rightarrow \eta_{\alpha\beta}$ and $G_{\alpha\beta} \rightarrow 0$ yielding to $-\Lambda\eta_{\alpha\beta} \sim 8\pi T_{\alpha\beta}$, where $\eta_{\alpha\beta} = \text{diag}(-1, 1, r^2, r^2 \sin^2 \theta)$ is the flat spacetime metric. Thus, we see that the de Sitter and anti-de Sitter spacetimes will be dominated in the $r \rightarrow \infty$ limit. Indeed, there are two inner and outer spacetimes like as the ones in the cut and paste procedure. But in this work and in contrast to the cut and paste procedure, an inner asymptotically vanishing geometry (asymptotically flat wormhole) is smoothly switching to the outer de Sitter or anti-de Sitter spacetime. Also, One can only consider one metric by imposing the asymptotically de Sitter or anti-de Sitter conditions on the metric and use $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ as the Einstein equation. This situation is like what we have in the Schwarzschild-de Sitter spacetime [51], which is also different than our approach in this work.

Finally, using the shape function, equation (13), we can express the third wormhole constraint mentioned in section II, by the following inequality

$$H(x, a, \alpha) \equiv ax^{-\alpha} - x^{-1} + 1 - a < 0, \quad (31)$$

where we defined $x \equiv \frac{r_0}{r}$. In order to cover the entire spacetime, the x has the range of $0 < x \leq 1$ where $x = 1$ corresponds to the wormhole throat, $r = r_0$, and $x \rightarrow 0$ corresponds to spatial infinity. In order to check this condition, we shall plot $H(x, a, \alpha)$ versus $x \equiv \frac{r_0}{r}$ for some specific choices of a and α at the end of the next section. Also, with note to the importance of the amount of energy violating matter, we will classify our obtained exact solutions to the wormholes with an unbounded or a bounded mass function. For the wormholes with a bounded mass function, a finite amount of energy conditions violating matter is sufficient in order to support the corresponding asymptotically flat wormhole geometry.

IV. SPECIFIC WORMHOLES WITH AN UNBOUNDED MASS FUNCTION

A. The case $a = 1$ and $\alpha = \frac{1}{2}$

This case is allowed for both $\Lambda > 0$ and $\Lambda < 0$. For this specific case, equation (23) can be solved:

$$U(r) = U_1 \exp \left[-\frac{\Lambda \left(3r^2 + 6r_0 r + 4r_0^2 \left(\frac{r}{r_0} \right)^{\frac{1}{2}} + 12r_0^2 \left(\frac{r}{r_0} \right)^{\frac{1}{2}} + 6r_0^2 \ln \left(\frac{r}{r_0} \right) \right)}{12\Lambda r_0^2 + 6} \right], \quad (32)$$

where we can absorb the constant U_1 into the re-scaled time coordinate. The mass function takes the simple form

$$m(r) = \frac{r_0}{2} \left[\left(\frac{r}{r_0} \right)^{\frac{1}{2}} - 1 \right] + \frac{\Lambda}{6} (r^3 - r_0^3), \quad (33)$$

and the pressures will be

$$\begin{aligned} p_r(r) &= \omega\rho(r) = \frac{\omega}{8\pi} \left[\frac{1}{2r_0^2} \left(\frac{r_0}{r} \right)^{\frac{5}{2}} + \Lambda \right], \\ p_t(r) &= -\frac{1}{4}p_r + \frac{5\Lambda\omega}{32\pi} + \frac{8p_r \left(\frac{\omega+2}{\omega} \right) \pi r^2 - \Lambda r^2}{8(\Lambda r_0^2 + 1) \left(1 - \frac{16\pi}{\omega} p_r r^2 + 2\Lambda r^2 \right)} p_r. \end{aligned} \quad (34)$$

Also, by considering the constant redshift function $U(r) = 1$, we recover $m(r)$ in equation (33) and the pressures will be

$$\begin{aligned} p_r(r) &= -\frac{1}{8\pi} \left[\frac{r_0}{r^3} \sqrt{\frac{r}{r_0}} + \Lambda \right], \\ p_l(r) &= \frac{1}{8\pi} \left[\frac{1}{4r^2} \sqrt{\frac{r_0}{r}} - \Lambda \right], \end{aligned} \quad (35)$$

which have the asymptotics as $r \rightarrow \infty$ as

$$\begin{aligned} p_r(r) &\rightarrow -\frac{\Lambda}{8\pi}, \\ p_l(r) &\rightarrow -\frac{\Lambda}{8\pi}, \end{aligned} \quad (36)$$

corresponding to the asymptotically de Sitter or anti-de Sitter spacetimes.

B. The case $a = 1$ and $\alpha = \frac{1}{3}$

This case is also allowed for both $\Lambda > 0$ and $\Lambda < 0$. For the specific case $a = 1$ and $\alpha = \frac{1}{3}$, solving the equation (23) gives the solution

$$U(r) = U_2 \exp \left[-\frac{\Lambda \left(2r^2 + 3r_0 r \left(\frac{r}{r_0} \right)^{\frac{1}{3}} + 6r_0^2 \left(\frac{r}{r_0} \right)^{\frac{2}{3}} + 4r_0^2 \ln \left(\frac{r}{r_0} \right) \right)}{6\Lambda r_0^2 + 2} \right], \quad (37)$$

where we can treat U_2 as previous section.

Also, for the mass function, we will have

$$m(r) = \frac{r_0}{2} \left[\left(\frac{r}{r_0} \right)^{\frac{1}{3}} - 1 \right] + \frac{\Lambda}{6} (r^3 - r_0^3). \quad (38)$$

The pressures also will be

$$\begin{aligned} p_r(r) &= \omega \rho(r) = \frac{\omega}{8\pi} \left[\frac{1}{3r_0^2} \left(\frac{r_0}{r} \right)^{\frac{8}{3}} + \Lambda \right], \\ p_l(r) &= -\frac{1}{3} p_r + \frac{\Lambda \omega}{6\pi} + \frac{4\pi p_r \left(\frac{\omega+3}{\omega} \right) r^2 - \Lambda r^2}{3(\Lambda r_0^2 - 1) \left(1 - \frac{24\pi}{\omega} p_r r^2 + 3\Lambda r^2 \right)} p_r. \end{aligned} \quad (39)$$

Also, by considering the constant redshift function $U(r) = 1$, we recover $m(r)$ in equation (33) and the pressures will be

$$\begin{aligned} p_r(r) &= -\frac{1}{8\pi} \left[\frac{r_0}{r^3} \left(\frac{r}{r_0} \right)^{\frac{1}{3}} + \Lambda \right], \\ p_l(r) &= \frac{1}{8\pi} \left[\frac{1}{3r^2} \left(\frac{r_0}{r} \right)^{\frac{2}{3}} - \Lambda \right], \end{aligned} \quad (40)$$

which have the asymptotics as $r \rightarrow \infty$ as

$$\begin{aligned} p_r(r) &\rightarrow -\frac{\Lambda}{8\pi}, \\ p_l(r) &\rightarrow -\frac{\Lambda}{8\pi}, \end{aligned} \quad (41)$$

corresponding to the asymptotically de Sitter or anti-de Sitter spacetimes.

C. The case $a = \frac{1}{2}$ and $\alpha = \frac{1}{2}$

This case is also acceptable for $\Lambda > 0$ and $\Lambda < 0$. For this configuration, one obtains the following solution

$$U(r) = U_3 \exp \left[- \frac{\Lambda \left(12r^2 + 18r_0r + 8r_0r \left(\frac{r}{r_0} \right)^{\frac{1}{2}} + 30r_0^2 \left(\frac{r}{r_0} \right)^{\frac{1}{2}} + 32r_0^2 \ln \left(\frac{r}{r_0} \right) - 31r_0^2 \ln \left(2 \left(\frac{r}{r_0} \right)^{\frac{1}{2}} + 1 \right) \right)}{32\Lambda r_0^2 + 8} \right] \\ \times \left[4 - 4\sqrt{\frac{r_0}{r}} - \frac{r_0}{r} \right]^{\frac{1}{4\Lambda r_0^2}}, \quad (42)$$

where again, absorption of the constant U_3 would be applicable into the re-scaled time coordinate.

The mass function will be written as

$$m(r) = \frac{r_0}{4} \left[\left(\frac{r}{r_0} \right)^{\frac{1}{2}} - 1 \right] + \frac{\Lambda}{6} (r^3 - r_0^3), \quad (43)$$

while the pressures are obtained as

$$p_r(r) = \omega \rho(r) = \frac{\omega}{8\pi} \left[\frac{1}{4r_0^2} \left(\frac{r_0}{r} \right)^{\frac{5}{2}} + \Lambda \right], \\ p_l(r) = -\frac{1}{4}p_r + \frac{5\Lambda\omega}{32\pi} + \frac{24\pi p_r \left(\frac{\omega+2}{\omega} \right) r^2 - 3\Lambda r^2 + \frac{3r_0}{2r}}{16(\Lambda r_0^2 + 1) \left(1 - \frac{16\pi}{\omega} p_r r^2 + 2\Lambda r^2 - \frac{r_0}{2r} \right)} p_r. \quad (44)$$

Also, by considering the constant redshift function $U(r) = 1$, we recover $m(r)$ in equation (33) and the pressures will be

$$p_r(r) = -\frac{1}{8\pi} \left[\frac{r_0}{2r^3} \left(1 + \left(\frac{r}{r_0} \right)^{\frac{1}{2}} \right) + \Lambda \right], \\ p_l(r) = \frac{1}{8\pi} \left[\frac{r_0}{8r^3} \left(2 + \left(\frac{r}{r_0} \right)^{\frac{1}{2}} \right) - \Lambda \right], \quad (45)$$

which have the asymptotics as $r \rightarrow \infty$ as

$$p_r(r) \rightarrow -\frac{\Lambda}{8\pi}, \\ p_l(r) \rightarrow -\frac{\Lambda}{8\pi}, \quad (46)$$

corresponding to the asymptotically de Sitter or anti-de Sitter spacetimes.

D. The case $a = -1$ and $\alpha = \frac{1}{2}$

This case is only allowed for $\Lambda > 0$. Applying the values $a = -1$ and $\alpha = \frac{1}{2}$, gives a solution for the equation (23) as

$$U(r) = U_5 \times \exp \left[- \frac{\Lambda \left(3r^2 - 60r_0^2 \left(\frac{r}{r_0} \right)^{\frac{1}{2}} + 18r_0r + 124r_0^2 \ln \left(\left(\frac{r}{r_0} \right)^{\frac{1}{2}} + 2 \right) + 4r_0^2 \ln \left(\frac{r}{r_0} \right) - 4r_0r \left(\frac{r}{r_0} \right)^{\frac{1}{2}} - 2 \ln \left(\frac{r}{r_0} \right) + 4 \ln \left(\left(\frac{r}{r_0} \right)^{\frac{1}{2}} + 2 \right) \right)}{4\Lambda r_0^2 - 2} \right], \quad (47)$$

where the constant U_4 can be absorbed similar to the previous solutions.

The mass function would be written as

$$m(r) = \frac{r_0}{2} \left[1 - \left(\frac{r}{r_0} \right)^{\frac{1}{2}} \right] + \frac{\Lambda}{6} (r^3 - r_0^3), \quad (48)$$

and the pressures will be

$$\begin{aligned} p_r(r) &= \omega\rho(r) = \frac{\omega}{8\pi} \left[-\frac{1}{2r_0^2} \left(\frac{r_0}{r} \right)^{\frac{5}{2}} + \Lambda \right], \\ p_l(r) &= -\frac{1}{4}p_r + \frac{5\Lambda\omega}{32\pi} + \frac{24\pi p_r \left(\frac{\omega+2}{\omega} \right) r^2 - 3\Lambda r^2 + 6\frac{r_0}{r}}{8(\Lambda r_0^2 + 1) \left(1 + \frac{16\pi}{\omega} p_r r^2 + 2\Lambda r^2 - 2\frac{r_0}{r} \right)} p_r. \end{aligned} \quad (49)$$

Also, by considering the constant redshift function $U(r) = 1$, we recover $m(r)$ in equation (33) and the pressures will be

$$\begin{aligned} p_r(r) &= -\frac{1}{8\pi} \left[\frac{r_0}{r^3} \left(2 - \left(\frac{r}{r_0} \right)^{\frac{1}{2}} \right) + \Lambda \right], \\ p_l(r) &= \frac{1}{8\pi} \left[\frac{r_0}{4r^3} \left(4 - \left(\frac{r}{r_0} \right)^{\frac{1}{2}} \right) - \Lambda \right], \end{aligned} \quad (50)$$

which have the asymptotics as $r \rightarrow \infty$ as

$$\begin{aligned} p_r(r) &\rightarrow -\frac{\Lambda}{8\pi}, \\ p_l(r) &\rightarrow -\frac{\Lambda}{8\pi}, \end{aligned} \quad (51)$$

corresponding to the asymptotically de Sitter or anti-de Sitter spacetimes.

V. SPECIFIC WORMHOLES WITH A BOUNDED MASS FUNCTION

A. The case $a = -\frac{1}{2}$ and $\alpha = -1$

This case is allowed for both $\Lambda > 0$ and $\Lambda < 0$. Applying the values $a = -\frac{1}{2}$ and $\alpha = -1$, gives a solution for equation (23) as

$$U(r) = U_4 \exp \left[-\frac{\Lambda (2r^2 + 6r_0 r - 9r_0^2 \ln(2r - r_0) + 16r_0^2 \ln(r)) - 12 \ln(2r - r_0) + 12 \ln(r)}{8\Lambda r_0^2 + 4} \right], \quad (52)$$

where the constant U_4 can be absorbed similar to the previous solutions.

The mass function would be written as

$$m(r) = \frac{r_0}{4} \left[1 - \frac{r_0}{r} \right] + \frac{\Lambda}{6} (r^3 - r_0^3). \quad (53)$$

The pressures will be

$$\begin{aligned} p_r(r) &= \omega\rho(r) = \frac{\omega}{8\pi} \left[\frac{1}{2r_0^2} \left(\frac{r_0}{r} \right)^4 + \Lambda \right], \\ p_l(r) &= -p_r + \frac{\Lambda\omega}{4\pi} + \frac{8\pi p_r \left(\frac{\omega-1}{\omega} \right) r^2 + 2\Lambda r^2 + \frac{3r_0}{2r}}{8(\Lambda r_0^2 + 1) \left(1 + \frac{8\pi}{\omega} p_r r^2 - \Lambda r^2 - \frac{3r_0}{2r} \right)} p_r. \end{aligned} \quad (54)$$

Also, by considering the constant redshift function $U(r) = 1$, we recover $m(r)$ in equation (33) and the pressures will be

$$\begin{aligned} p_r(r) &= -\frac{1}{8\pi} \left[\frac{r_0}{2r^4} (3r - r_0) + \Lambda \right], \\ p_l(r) &= \frac{1}{8\pi} \left[\frac{r_0}{4r^4} (3r - 2r_0) - \Lambda \right], \end{aligned} \quad (55)$$

which have the asymptotics as $r \rightarrow \infty$ as

$$\begin{aligned} p_r(r) &\rightarrow -\frac{\Lambda}{8\pi}, \\ p_l(r) &\rightarrow -\frac{\Lambda}{8\pi}, \end{aligned} \quad (56)$$

corresponding to the asymptotically de Sitter or anti-de Sitter spacetimes.

B. The case $a = \frac{1}{2}$ and $\alpha = -1$

This case is only allowed for $\Lambda > 0$. For the specific case $a = \frac{1}{2}$ and $\alpha = -1$, solving the equation (23) gives the solution

$$U(r) = U_2 \exp \left[-\frac{\Lambda (6r^2 + 6r_0 r - 7r_0^2 \ln(2r + r_0) + 16r_0^2 \ln(r)) - 4 \ln(2r + r_0) + 4 \ln(r)}{8\Lambda r_0^2 - 4} \right], \quad (57)$$

where we can treat U_2 as previous sections.

For the mass function, we have

$$m(r) = \frac{r_0}{4} \left[\left(\frac{r_0}{r} \right) - 1 \right] + \frac{\Lambda}{6} (r^3 - r_0^3). \quad (58)$$

The pressures will be

$$\begin{aligned} p_r(r) &= \omega \rho(r) = \frac{\omega}{8\pi} \left[-\frac{1}{2r_0^2} \left(\frac{r_0}{r} \right)^4 + \Lambda \right], \\ p_l(r) &= -p_r + \frac{\Lambda \omega}{4\pi} + \frac{24\pi p_r \left(\frac{\omega-1}{\omega} \right) r^2 + 6\Lambda r^2 + \frac{3r_0}{2r}}{8(\Lambda r_0^2 + 1) \left(1 + \frac{8\pi}{\omega} p_r r^2 - \Lambda r^2 - \frac{r_0}{2r} \right)} p_r. \end{aligned} \quad (59)$$

Also, by considering the constant redshift function $U(r) = 1$, we recover $m(r)$ in equation (33) and the pressures will be

$$\begin{aligned} p_r(r) &= -\frac{1}{8\pi} \left[\frac{r_0}{2r^4} (r + r_0) + \Lambda \right], \\ p_l(r) &= \frac{1}{8\pi} \left[\frac{r_0}{4r^4} (r + 2r_0) - \Lambda \right], \end{aligned} \quad (60)$$

which have the asymptotics as $r \rightarrow \infty$ as

$$\begin{aligned} p_r(r) &\rightarrow -\frac{\Lambda}{8\pi}, \\ p_l(r) &\rightarrow -\frac{\Lambda}{8\pi}, \end{aligned} \quad (61)$$

corresponding to the asymptotically de Sitter or anti-de Sitter spacetimes.

The corresponding $H(x, a, \alpha)$ function for all of these cases are shown in the following figure, Figure 1.

As seen from the figure, the function $H(x, a, \alpha)$ is negative for all cases throughout the entire range of x , indicating the satisfaction of the third wormhole condition.

Also, it is interesting to evaluate the "volume integral quantifier" [49, 50] which provides information about the "total amount" of energy condition violating matter. This quantity is given by

$$I_V \equiv \oint [\rho(r) + p_r(r)] dV = 2 \int_{r_0}^{\infty} [\rho(r) + p_r(r)] 4\pi r^2 dr, \quad (62)$$

which by considering the equation (18) and the equation of state $p_r(r) = \omega \rho(r)$ gives the solution as

$$I_V = (\omega + 1) \left[a r_0 \left(\frac{r}{r_0} \right)^\alpha + \frac{1}{3} \Lambda r^3 \right] \Big|_{r_0}^{\infty}. \quad (63)$$

This equation shows that in the absence of cosmological constant, in order to have a finite amount of "energy condition violating matter", the α values must be negative, in which case we have

$$I_V \rightarrow -(\omega + 1) a r_0. \quad (64)$$

Therefore, as $a \rightarrow 0$, we have $I_V \rightarrow 0$ which reflects arbitrary small quantities of energy condition violating matter. In the presence of cosmological constant, it is seen that the sign of equation (50) is not fixed and depends on the parameters of the model and cosmological constant.

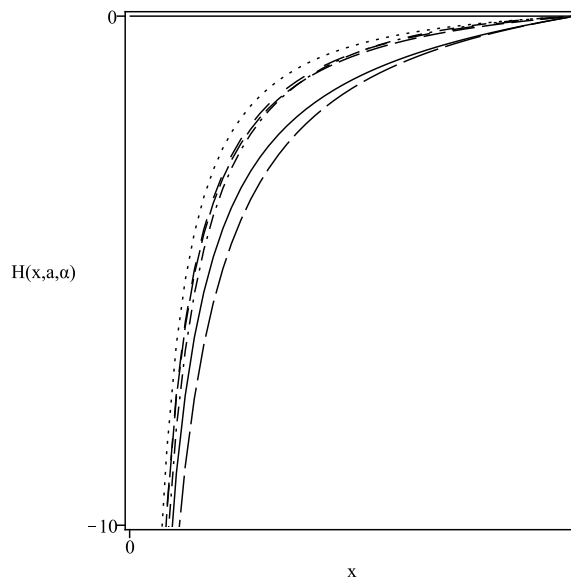


FIG. 1: The plots depict $H(x, a, \alpha)$ function in which the solid, dot, spacedash, dash, longdash and dashdot plots stand for the $H(x, 1, 1/2)$, $H(x, 1, 1/3)$, $H(x, 1/2, 1/2)$, $H(x, -1, 1/2)$, $H(x, -1/2, -1)$ and $H(x, 1/2, -1)$, respectively. The parameter $x = r_0/r$, lying in the range $0 < x \leq 1$, has been defined in order to cover the entire spacetime.

VI. CONCLUDING REMARKS

The asymptotically flat wormhole solutions embedded in a generic cosmological constant background rather than a vacuum spacetime are investigated. Indeed, there are two inner and outer spacetimes like as the ones in the cut and paste procedure. But in this work and in contrast to the cut and paste procedure, an inner asymptotically vanishing geometry (asymptotically flat wormhole) is smoothly switching to the outer de Sitter or anti-de Sitter spacetime without need to a surgically pasting. Near the throat, the wormhole geometry is dominant while as the radii increases the wormhole features disappear and the characterizations of the cosmological constant will be more clarified, signalling us that our solutions include an asymptotic spacetime, the background, which should normally be de Sitter or anti-de Sitter, depending on the positive or negative values of the cosmological constant. It is shown that for constructing such a geometry, the wormhole throat r_0 , cosmological constant Λ and the equation of state parameter ω should satisfy certain relations. With respect to the sign of Λ , corresponding to de Sitter or anti-de Sitter spacetime, a new restricting condition on wormhole throat size is obtained. Also, it is shown that in the presence of cosmological constant Λ with a general redshift function $U(r)$, the energy density profile $\rho(r)$, the radial pressure $p_r(r)$ and the lateral pressure $p_l(r)$ take an asymptotically de Sitter or anti de Sitter behavior at $r \rightarrow \infty$. Also, it is denoted that these asymptotics are simply achieved by choosing a constant redshift function. Then, using the possibility of setting different values for the parameters of the model, some exact solutions leading to specific metrics, mass functions and supporting energy momentum profiles are found. The volume integral quantifier, which provides useful information about the total amount of energy condition violating matter is also briefly mentioned. It is shown that the amount of this energy condition violation depends on the parameters of the model and the value of cosmological constant which can be fixed from observations.

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